Watson’s Lemma with Applications

by

Khaled Muhammad Al-Furati

A Thesis Presented to the

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ABSTRACT

This thesis is about Watson's Lemma. This lemma describes the behavior of the Laplace and Laplace-Stieltjes integrals near infinity. Consequently, we can use this lemma to find asymptotic expansions of functions that can be written in the Laplace or Laplace-Stieltjes integral form. This study is mainly based on the Riemann-Stieltjes and Laplace-Stieltjes integrals.

" الخلاصة "

نتيجة واتسون هى نتيجة تصف السلوك المقارب لتكامل لابلاس وتكامل لابلاس ستيلتجس عند اللانهاية ، وبالتالى يمكننا الحصول على مفكوك مقارب لبعض الدوال التى يمكن كتابتها على هيئة هذين التكاملين . هذه الدراسة ترتكز على تكامل ريمان ستيلتجس ولابلاس ستيلتجس .

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INTRODUCTION

Watson's Lemma is a result which describes the asymptotic behavior of the Laplace and Laplace-Stieltjes integrals near infinity. It shows that this behavior depends mainly on the asymptotic behavior of the determining function near the origin. In the next few paragraphs we provide a brief outline of this thesis.

In the first chapter we give a brief study of the Riemann-Stieltjes integral. In section 1 we define this integral. In section 2 we summarize the concept of bounded variation which is an essential condition for existence. Section 3 contains some theorems which give some sufficient conditions for the existence of the Riemann-Stieltjes integral. This chapter is concluded by section 4 which contains some of the useful properties of the Riemann-Stieltjes integral.

The second chapter is devoted to the study of the Laplace-Stieltjes integral. The definition of this integral is given in section 1 and in section 2 we discuss the three types of convergence of this integral : simple, absolute and uniform convergence. Set of convergence of each type is determined. In section 3 we show that the Laplace-Stieltjes integral represents an analytic function in the set of uniform convergence. In section 4 we show that not only it is analytic but, in addition, it behaves well as it approaches the finite boundary points of the set of simple convergence which are in the set.

In the third chapter we study asymptotic expansions. In section 1 we define Landau symbols and in section 2 we define asymptotic power series and give some illustrative examples. In section 3 we study some operations on asymptotic power series and in section 4 we look at asymptotic power series of analytic functions. Finally, in section 5 we introduce the definition of asymptotic developments, which are more general than asymptotic power series.

The last chapter, chapter 4 , is devoted to, the study of the asymptotic behavior of the Laplace-Stieltjes integral. In section 1 we show the relation between the order of the determining function and the convergence of its transform. We use this relation to express the Laplace-Stieltjes integral in terms of the Laplace integral. In section 2 we determine the value of the Laplace-Stieltjes integral at the origin and infinity. In section 3 and 4 we prove Watson's Lemma for the Laplace-Stieltjes and Laplace integrals and how can be used to get asymptotic expansions. Some examples are presented in section 5 . Finally, in section 6 we illustrate two examples which show the power of asymptotic expansions even if they diverge.

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Chapter I

THE RIEMANN-STIELTJES INTEGRAL

In this chapter we deal with the Riemann-Stieltjes integral. Some of its properties , which will be needed later for the study of the Laplace-Stieltjes integral , are given. the significance of the Riemann-Stieltjes integral for our purpose is due to two features : First, it contains the Riemann integral , definite and indefinite. Second, it holds integration by parts , which is an essential tool in many proofs .

1.1 Definition

Let α and f be real valued functions defined on $[a,b]$. Let P denotes a partition of $[a,b]$ by the points $x_0, x_1, \dots, x_n$, where $a=x_0<x_1<\dots<x_n=b$. Let $\delta = \max\limits_{0\le i\le n} \{x_{i+1}-x_i\}$. We call δ the norm of P , $|P|$.

Definition(1.1.1). If the limit

$$\lim_{\delta\to 0}\sum_{i=0}^{n-1} f(z_i)\,[\,\alpha(x_{i+1})-\alpha(x_i)\,]\,,$$

where, $x_i \le z_i \le x_{i+1}$, exists independently of the partition and the choice of z_i , then the limit is called the **Riemann-Stieltjes Integral of** f **with respect to** α over $[a,b]$ and is denoted by

$$\int_a^b f\,d\alpha \quad or \quad \int_a^b f(x)\,d\alpha\,(x)\,,$$

and f is said to be **integrable with respect to** α .

The definition can be easily extended to include complex functions. If $f(x) = f_1(x) + if_2(x)$ and $\alpha(x) = \alpha_1(x) + i\alpha_2(x)$, where f_1, f_2, α_1 and α_2 are real valued functions , we define

$$\int_a^b f\,d\alpha = \int_a^b f_1\,d\alpha_1 - \int_a^b f_2\,d\alpha_2 + i\left\{\int_a^b f_1\,d\alpha_2 + \int_a^b f_2\,d\alpha_1\right\},$$

provided all the integrals on the right exist .

Remark(1.1.2). If $\alpha(x) = x$ and f is a Riemann integrable function , then the integral reduces to the Riemann integral. This is one reason why the Riemann-Stieltjes integral bears its name .

Notation. Throughout we adopt the following notations :

R : The set of all Riemann integrable functions .

$R(\alpha)$: The set of all Riemann-Stieltjes integrable functions with respect to α .

RSI : The Riemann-Stieltjes integral.

1.2 Functions of Bounded Variation

Definition(1.2.1). A function α defined on $[a,b]$ is of **bounded variation in** $[a,b]$ if there exists a constant $M > 0$ such that for any partition $P = \{ a = x_0 < x_1 < \ldots < x_n = b \}$ of $[a,b]$

$$v(\alpha, P) = \sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})| \leq M .$$

The **total variation** of α , $V(\alpha)$, is defined by

$$V(\alpha) = \sup \{ v(\alpha, P) : P \text{ is a partition of } [a,b] \} .$$

Clearly $V(\alpha) \leq M < \infty$.

Remark(1.2.2). We have two remarks :

(1) Every monotonic function is of bounded variation and $V(\alpha) = |\alpha(b) - \alpha(a)|$.

(2) If α is a complex function , then $V(\alpha)$ is the length of the curve $\alpha : \alpha(t)\, , a \le t \le b$. So, if α is of bounded variation , then the length of α is called rectifiable .

1.3 Existence of RSI

In this section we consider sufficient conditions on α and f for the existence of the RSI $\int_a^b f\, d\alpha$.

Theorem(1.3.1). If f is continuous in $[a,b]$ and α is of bounded variation in $[a,b]$, then $f \in R(\alpha)$.

For the proof of this theorem see [19;*P*.7] and [1;*P*.60] .

The following theorem shows an important property of the RSI which is integration by parts.

Theorem(1.3.2). If f is of bounded variation in $[a,b]$ and α is continuous in $[a,b]$, then $\alpha \in R(f)$ and

$$\int_a^b f(x)\, d\alpha(x) = f(b)\,\alpha(b) - f(a)\,\alpha(a) - \int_a^b \alpha(x)\, df(x) \ .$$

Proof. Choose an arbitrary partition $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ such that $|p| = \delta$, and form the sum

$$S_P = \sum_{k=0}^{n-1} \alpha(z_k)\,[\, f(x_{k+1}) - f(x_k)\,] \ , \quad x_k \le z_k \le x_{k+1} \ .$$

By use of partial summation we have

$$S_P = -\sum_{k=1}^{n-1} f(x_k)\,[\,\alpha(z_k) - \alpha(z_{k-1})\,] - f(a)\,[\,\alpha(z_0) - \alpha(a)\,] - f(b)\,[\,\alpha(b) - \alpha(z_{n-1})\,]$$

$$+ f(b)\,\alpha(b) - f(a)\,\alpha(a) \ .$$

Then ,

$$\lim_{\delta \to 0} S_P = -\int_a^b f(x)\, d\alpha(x) + f(b)\,\alpha(b) - f(a)\,\alpha(a) .$$

This proves the theorem .

1.4 Properties of the RSI

In this section we collect certain useful properties of the RSI .

Theorem(1.4.1). Let α be any real valued function . Then ,

(a) $\int_a^b d\alpha(x) = \alpha(b) - \alpha(a)$.

(b) If $\alpha(x)$ is constant , then $\int_a^b f(x)\, d\alpha(x) = 0$.

(c) If $f_1, f_2 \in R(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in R(\alpha)$ and

$$\int_a^b [f_1 + f_2]\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha .$$

(d) If $f \in R(\alpha)$ on $[a,b]$, then $cf \in R(\alpha)$ for every constant c and

$$\int_a^b c f\, d\alpha = c \int_a^b f\, d\alpha .$$

(e) If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$, then $f \in R(\alpha_1 + \alpha_2)$ and

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 .$$

(f) If $f \in R(\alpha)$ on $[a,b]$ and if $a < c < b$, then $f \in R(\alpha)$ on $[a,c]$ and on $[c,b]$ and

$$\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha .$$

Proof. The proofs follow directly from the definition just by rearranging the summation. In part (f) the point is that we may restrict ourselves to partitions which contain the point c .

Theorem(1.4.2). If f is continuous on $[a,b]$ and α is of bounded variation in $[a,b]$, then

$$\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,|d\alpha|$$

For the proof see [19;P.8] .

Theorem(1.4.3). Let α be of bounded variation on $[a,b]$ and $\alpha' \in R$ on $[a,b]$. Let f be a continuous real function on $[a,b]$. Then $f\alpha' \in R$ and

$$\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx .$$

For the proof see [1 ; P.62] .

Theorem(1.4.4). If f is a continuous function in $[a,b]$ and α is of bounded variation in $[a,b]$, then

$$F(x) = \int_a^x f\,d\alpha$$

is also of bounded variation in $[a,b]$ and

$$V_F(x) = \int_a^x |f|\,|d\alpha| .$$

Theorem(1.4.5). If f and φ are continuous and $\alpha(x)$ is of bounded variation on $[a,b]$, and if

$$\beta(x) = \int_c^x \varphi(t)\,d\alpha(t) ,$$

then

$$\int_a^b f\,d\beta = \int_a^b f\varphi\,d\alpha .$$

We conclude this section by stating the following theorem , which is about change of variables.

Theorem(1.4.6). Suppose φ is strictly increasing continuous function that maps an interval $[A,B]$ on $[a,b]$. Define β and g on $[A,B]$ by $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$. Then , $g \in R(\beta)$ and

$$\int_A^B g\,d\beta = \int_a^b f\,d\alpha \ .$$

For the proof see [1;P.132] .

Now we investigate the extent to which the definition of f or α can be changed without affecting the value of $\int_a^b f\,d\alpha$.

Theorem(1.4.7). If f is continuous on $[a,b]$, E is a dense subset of $[a,b]$ that contains a and b , and α is of bounded variation on $[a,b]$ which is constant on E , then

$$\int_a^b f\,d\alpha = 0$$

Proof. The limit in Definition (1.1.1) exists independently of the manner the subdivision of $[a,b]$ is chosen . Hence we may choose x_k in E so the limit is zero.

Corollary(1.4.8). If α_1 and α_2 are of bounded variation on $[a,b]$ and differ by a constant c on on the set E of Theorem (1.4.7) and if f is continuous on $[a,b]$, then

$$\int_a^b f\,d\alpha_1 = \int_a^b f\,d\alpha_2$$

Proof. Follows by applying Theorem (1.4.7) on $\alpha = \alpha_1 - \alpha_2 = c$.

Chapter II

THE LAPLACE-STIELTJES INTEGRAL

In this chapter we develop some of the most useful properties of the Laplace-Stieltjes integral : convergence , analyticity and the behavior at the finite boundary points of the set of convergence.

2.1 Definition

Suppose that α is a real valued function on $[0,\infty)$ which is of bounded variation on every interval $[0,R]$, $R>0$. Then the **Laplace-Stieltjes integral** ,**LSI,** of α is defined as the function

$$f(s) = \int_0^\infty e^{-st}\, d\alpha(t)$$

for all complex s for which the integral converges .

The integral on the right is improper of the upper limit and to be understood as

$$f(s) = \lim_{R\to\infty} \int_0^R e^{-st}\, d\alpha(t) \ .$$

If

$$\varphi(s) = \int_0^\infty e^{-st}\, \psi(t)\, dt \ ,$$

we refer to φ as the **Laplace Integral** of ψ . Also f is called the **generating function,** whereas ψ , and sometimes α , are called the **determining functions** .

We assume throughout, unless otherwise stated, that α is of bounded variation on $[0,\infty)$.

2.2 Set of Convergence

In this section we consider three types of sets of convergence : simple , absolute and uniform convergence of the integral $f(s)$.

I. Simple Convergence

Definition(2.2.1). The set of all s such that $f(s)$ converges is called the set of simple convergence of $f(s)$ and is denoted by $C(\alpha)$.

Throughout , s and s_o are complex numbers , such that $\operatorname{Re} s = \sigma$ and $\operatorname{Re} s_o = \sigma_o$.

Theorem(2.2.2). Let $\beta(u) = \int_0^u e^{-s_o t}\, d\alpha(t)$ and suppose that

$$\sup_{0 \le u < \infty} |\beta(u)| \le M < \infty \, . \qquad (*)$$

Then $f(s)$ converges for all s with $\sigma > \sigma_o$, and

$$\int_0^\infty e^{-st}\, d\alpha(t) = (s - s_o) \int_0^\infty e^{-(s-s_o)t} \beta(t)\, dt \, ,$$

where the integral on the right-hand side converges absolutely.

Proof. By Theorem (1.4.5),

$$\int_0^R e^{-st}\, d\alpha(t) = \int_0^R e^{-(s-s_o)t}\, d\beta(t) \, .$$

Integration by parts gives

$$\int_0^R e^{-(s-s_o)t}\, d\beta(t) = e^{-(s-s_o)R} \beta(R) + (s - s_o) \int_0^R e^{-(s-s_o)t} \beta(t)\, dt \, .$$

Since β is uniformly bounded ,

$$\lim_{R \to \infty} e^{-(s-s_o)R} \beta(R) = 0 \, ,$$

for $\sigma > \sigma_o$. Therefore, $f(s)$ converges for s such that $\operatorname{Re} s > \sigma_o$, and equality holds . For the second part of the theorem we have for x and y , $x < y$,

$$\int_x^y |e^{-(s-s_o)t} \beta(t)|\, dt \le M \int_x^y e^{-(\sigma-\sigma_o)t} dt = \frac{M}{\sigma_o - \sigma} e^{-(\sigma-\sigma_o)t} \Big|_x^y$$

$$= \frac{M}{\sigma_o - \sigma} (e^{-y} - e^{-x})^{-(\sigma-\sigma_o)}$$

which tends to zero as x and y tend to infinity. This completes the proof.

Note that condition (*) in Theorem (2.2.2) does not imply the convergence of $f(s)$ at s_o . For example, If $\alpha(t) = \cos t$, then condition (*) is satisfied for $s_o = 0$ but the LSI of α does not converge. However the converse is true .

Corollary(2.2.3). If $f(s)$ converges for s_o , then it converges for all s , $\operatorname{Re} s > \operatorname{Re} s_o$.

As a consequence we have :

Theorem(2.2.4). The set of simple convergence is either empty , the full plane or a right-half plane , possibly including some or all of its boundary points .

In view of this result we introduce

Definition(2.2.5). The abscissa of simple convergence of $f(s)$ is the value

$$\sigma_c = \begin{cases} \inf\{\operatorname{Re} s : s \in C(\alpha)\} & \text{if } C(\alpha) \text{ is not empty}, \\ \infty & \text{otherwise}. \end{cases}$$

Note that if $C(\alpha)$ is the whole complex plane , then $\sigma_c = -\infty$, and if $\sigma_c < \infty$, then $C(\alpha)$ contains all s such that $\operatorname{Re} s > \sigma_c$ and no s such that $\operatorname{Re} s < \sigma_c$. No general statement can be said about the values s for which $\operatorname{Re} s = \sigma_c$.

Example(2.2.6).

(a) $\int_0^\infty e^{-st} e^{e^t}\, dt \qquad \sigma_c = \infty.$

(b) $\int_0^\infty e^{-st} e^{-e^t}\, dt \qquad \sigma_c = -\infty.$

(c) $\int_0^\infty e^{-st}\, dt \qquad \sigma_c = 0.$

II. Absolute Convergence

Definition(2.2.7). The integral $f(s) = \int_0^\infty e^{-st}\, d\alpha(t)$ is said to converge absolutely at s with $\operatorname{Re} s = \sigma$ if the integral

$$\int_0^\infty |e^{-st}|\,|d\alpha(t)| = \int_0^\infty e^{-\sigma t}\, |d\alpha(t)| = \int_0^\infty e^{-\sigma t}\, du(t). \qquad (**)$$

converges , where $u(x)$ is the total variation of α in $[0,x]$. In particular, if

$$\alpha(t) = \int_0^t \varphi(y)\, dy\ ,$$

then the total variation of α is

$$u(t) = \int_0^t |\varphi(y)|\, dy\ ,$$

and (**) becomes

$$\int_0^\infty e^{-\sigma t}\, |\varphi(t)|\, dt\ .$$

Let A(α) denotes the set of all s such that (**) converges.

Theorem(2.2.8). If $f(s)$ converges absolutely for s_o *with* $\operatorname{Re} s_o = \sigma_o$, then it converges absolutely in the half plane $\sigma \ge \sigma_o$.

Proof. We have for $\sigma \ge \sigma_o$

$$|e^{-st}\, d\alpha| = e^{-\sigma t}\, |d\alpha| \le e^{-\sigma_0 t}\, |d\alpha| = |e^{-s_0 t}\, d\alpha| \, .$$

Since $\int_0^\infty |e^{-s_0 t}\, d\alpha|$ converges, $\int_0^\infty |e^{-st}\, d\alpha|$ also converges. This completes the proof.

Theorem(2.2.8) implies the following

Theorem(2.2.9). The set of absolute convergence is either empty , the full plane or an open or closed right-half plane.

Now we can have the following definition

Definition(2.2.10). The **abscissa of absolute convergence** of $f(s)$ is the value

$$\sigma_a = \begin{cases} \inf\{\operatorname{Re} s : s \in A(\alpha)\} & \text{if } A(\alpha) \text{ is not empty}, \\ \infty & \text{otherwise}. \end{cases}$$

The following example shows that σ_c and σ_a need not be equal.

Example(2.2.11). Consider the integral

$$\int_0^\infty e^{-st} e^{kt} \sin e^{kt}\, dt, \qquad (k > 0)$$

The integral converges absolutely for $\sigma > k$ since $|e^{-st} e^{kt} \sin e^{kt}| \le e^{-(\sigma-k)t}$, $0 \le t < \infty$. But , it does not converge absolutely for $s = k$ since

$$\int_0^\infty |\sin e^{kt}|\, dt = \frac{1}{k} \int_1^\infty \frac{|\sin u|}{u}\, du = \infty \, .$$

Hence , $\sigma_a = k$. On the other hand, the integral

$$\int_0^\infty e^{-st} e^{kt} \sin e^{kt}\, dt = \frac{1}{k} \int_1^\infty \frac{\sin u}{u^{s/k}}\, du$$

converges for all $s < k$. Hence, $\sigma_c \ne \sigma_a$.

III. Uniform Convergence

Definition(2.2.12). The integral $f(s)$ is said to converge uniformly on a set E if for every $\varepsilon > 0$ there exists $w_o > 0$ such that for $w > w_o$

$$\left| \int_w^\infty e^{-st}\, d\alpha(t) \right| < \varepsilon$$

for all $s \in E$.

The following theorems provide us with sufficient conditions for uniform convergence.

Theorem(2.2.13). If $f(s)$ converges for s_o , then it converges uniformly on the set $S_\delta = \{ s : | Arg\,(s - s_o) | \le \delta \}$, where $0 \le \delta < \frac{\pi}{2}$.

Proof. Let $\beta(u) = -\int_u^\infty e^{-s_o t}\, d\alpha(t)$, then given any $\varepsilon > 0$ there exists w_o such that $| \beta(u) | < \varepsilon$ for $u > w_o$. Thus, for $w_o < w < v$ we get by integration by parts

$$\int_w^v e^{-st}\, d\alpha(t) = \int_w^v e^{-st} e^{s_o t}\, d\beta(t)$$

$$= e^{-(s - s_o)t} \beta(t) \Big|_w^v + (s - s_o) \int_w^v e^{-(s - s_o)t} \beta(t)\, dt$$

The first term on the right is bounded by 2ε . For the second term

$$\left| (s - s_o) \int_w^v e^{-(s - s_o)t} \beta(t)\, dt \right| < \varepsilon\, | s - s_o | \int_w^v e^{-\mathrm{Re}\,(s - s_o)t}\, dt < \frac{| s - s_o |}{\mathrm{Re}\,(s - s_o)}\, \varepsilon$$

If $| Arg\,(s - s_o) | \le \delta$, where $\delta < \frac{\pi}{2}$, then $\mathrm{Re}\,(s - s_o) \ge | s - s_o | \cos\delta$. Hence, we have

$$\left| \int_w^v e^{-st}\, d\alpha(t) \right| < \left(2 + \frac{1}{\cos\delta} \right) \varepsilon .$$

Since the bound is independent of s , the uniformity of convergence is proved.

Theorem(2.2.14). If $f(s)$ converges absolutely for s_o, then it converges uniformly in the half plane $\sigma \geq \sigma_o$.

Proof. By hypothesis, given $\varepsilon > 0$, there exists $w_o > 0$ such that $\int_w^\infty e^{-\sigma_o t} \, |d\alpha(t)| < \varepsilon$ for $w > w_o$. But, for s with $\operatorname{Re} s = \sigma > \sigma_o$, we have

$$\left| \int_w^\infty e^{-st} \, d\alpha(t) \right| \leq \int_w^\infty e^{-\sigma t} \, |d\alpha(t)| \leq \int_w^\infty e^{-\sigma_o t} \, |d\alpha(t)| < \varepsilon .$$

This proves the theorem.

2.3 *Analyticity*

Here we show that the LSI represents an analytic function in its region of uniform convergence. To prove this we need the following :

Lemma(2.3.1). If $0 \leq a \leq b$ then $h(s) = \int_a^b e^{-st} \, d\alpha(t)$ is entire, and

$$h^{(k)}(s) = (-1)^k \int_a^b e^{-st} \, t^k \, d\alpha(t) .$$

Proof. By the uniformity of the exponential series we have

$$h(s) = \int_a^b e^{-st} \, d\alpha(t) = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \int_a^b t^n \, d\alpha(t) .$$

Clearly, the series converges uniformly, so that its sum is entire and

$$h^{(k)}(s) = \sum_{n=k}^{\infty} \frac{s^{n-k}}{(n-k)!} \int_a^b (-t)^n \, d\alpha(t) = \int_a^b e^{-st} (-t)^k \, d\alpha(t) .$$

This completes the proof.

Theorem(2.3.2). The LSI $f(s) = \int_0^\infty e^{-st} \, d\alpha(t)$ is analytic for all s with $\operatorname{Re} s > \sigma_c$ and

$$f^{(k)}(s) = (-1)^k \int_0^\infty e^{-st} \, t^k \, d\alpha(t) .$$

Proof. Suppose $f(s)$ converges for s and $\operatorname{Re} s = \sigma > \sigma_c$. We can surround s by a circle K which lies in the the half plane $\sigma > \sigma_c$. By Theorem (2.2.13), $f(s)$ converges uniformly in K. If $f_n(s) = \int_0^n e^{-st}\, d\alpha(t)$, n = 1,2,..., then f_n converges to f uniformly. Hence , f is analytic in K . By Weierstrass Double Series Theorem (If $f_n \to f$ uniformly , then $f_n' \to f'$ uniformly), the result is obtained.

2.4 *Behavior at the Boundary*

In this section we discuss the behavior of $f(s)$ at the finite boundary points of $C(\alpha)$. The behavior at infinity will be discussed in section 4.2 .

Suppose $\sigma_c < \infty$. Then, according to Theorem (2.3.3), f is analytic in the interior of $C(\alpha)$, $int\, C(\alpha)$, and consequently

$$\lim_{s \to s_o} f(s) = f(s_o)$$

holds for all $s_o \in int\, C(\alpha)$. On the other hand, for the finite points of the boundary of $C(\alpha)$, $\partial C(\alpha)$, we have the following theorem .

Theorem(2.4.1). If $s_o \in \partial C(\alpha) \cap C(\alpha)$ and $s_o \neq \infty$, then for every β , $0 \leq \beta < \frac{\pi}{2}$, we have

$$\lim_{s \to s_o} f(s) = f(s_o)$$

provided that $s \to s_o$ in the angular domain $S_\beta = \{ s : | Arg(s - s_o) | \leq \beta \}$.

Proof. By Theorem (2.2.13), w can be selected such that $\left| \int_w^\infty e^{-st}\, d\alpha(t) \right| < \frac{\varepsilon}{3}$, for all $s \in \hat{S}_\beta = S_\beta \cup \{ s_o \}$, Let $f_w(s) = \int_0^w e^{-st}\, d\alpha(t)$. Then, we have

$$|f(s) - f(s_o)| \leq |f(s) - f_w(s)| + |f_w(s) - f_w(s_o)| + |f_w(s_o) - f(s_o)| < \frac{2\varepsilon}{3} + |f_w(s) - f_w(s_o)| .$$

By Lemma (2.3.1) f_w is entire , so there exists $\delta > 0$ such that $|s - s_o| < \delta$ implies that $|f_w(s) - f_w(s_o)| < \frac{\varepsilon}{3}$. Hence, for $|s - s_o| < \delta, s \in \hat{S}_\beta$, $|f(s) - f(s_o)| < \varepsilon$. This proves the theorem.

Chapter III

ASYMPTOTIC EXPANSIONS

Introduction.

The theory of asymptotics aims at describing the behavior of a function when a parameter, index or independent variable tends to a specific value. For example

$e^{-x} \approx 1 - x$ when x is small.

$\sin(x + \varepsilon) = \sin x + \varepsilon \cos x$, when ε is small.

Values of great importance are those near the boundary of the domain of definition of a function. For example, when dealing with an analytic function f defined on some unbounded region S of the complex plane, the behavior of $f(z)$ *as* $z \to \infty$ in S is of interest.

Another situation is that to relate a function f to a power series F which need not be convergent but still any partial sum F_n is a good approximation of f as z tends to a specific value. This idea is the motivation of defining the asymptotic power series, APS , later.

3.1 Landau Symbols

Definition(3.1.1). Let φ and ψ be defined in a common region S of the complex plane. Then we write

$$\varphi = O(\psi)$$

if there exists an $A > 0$ such that

$$|\varphi| \leq A|\psi|$$

for all $z \in S$.

This symbol has the following somewhat different use :

Definition(3.1.2). Let φ and ψ be defined in a common region S of the complex plane, and $z_o \in \overline{S}$, closure of S. Then we write

$$\varphi = O(\psi) \;\; as \;\; z \to z_o \;\; in \;\; S,$$

if

$$\lim_{z \to z_o,\, z \in S} \left| \frac{\varphi}{\psi} \right| < \infty .$$

Notice that z_o can be infinity.

We also introduce the following definition.

Definition(3.1.3). Let φ and ψ be defined in a common region S of the complex plane, and $z_o \in \overline{S}$, then we write

$$\varphi = o(\psi) \;\; as \;\; z \to z_o \;\; in \;\; S ,$$

if

$$\lim_{z \to z_o,\, z \in S} \left| \frac{\varphi}{\psi} \right| = 0 .$$

It is easy to verify the following :

Rules(3.1.4).

(i) If $\varphi(z) = o(z^n)$ as $z \to 0$, then also $\varphi(z) = O(z^n)$ as $z \to 0$.

(ii) If $\varphi(z) = O(z^{n+1})$ as $z \to 0$, then also $\varphi(z) = o(z^n)$ as $z \to 0$.

In (i) and (ii) the first representation is sharper

(iii) If m and n are integers

(a) $O(z^{-m}) + O(z^{-n}) = O(z^{-\min(m,n)})$ as $z \to \infty$.

(b) $O(z^{-m})\, O(z^{-n}) = O(z^{-m-n})$ as $z \to \infty$.

(iv) If $\varphi = O(\psi)$ and $\alpha > 0$, then $\varphi^{\alpha} = O(\psi^{\alpha})$.

(v) If $\varphi = O(\psi)$, $\psi = O(\chi)$, then $\varphi = O(\chi)$.

3.2 Asymptotic Power Series , APS

Definition(3.2.1). Let S be an unbounded set in the complex plane, f be a complex function defined on S and

$$F(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots$$

be a formal power series whose partial sums are denoted by

$$F_n = a_0 + a_1 z^{-1} + \dots + a_n z^{-n} .$$

Then F is called the **asymptotic power series**, APS , of f as $z \to \infty$ in S if one of the following properties is satisfied :

Property (A). For n = 0,1,2,... , we have

$$f(z) - F_n(z) = O(z^{-n-1}) \quad as \quad z \to \infty \quad in \quad S .$$

That is there exist positive real numbers γ_n and ρ_n such that

$$|f(z) - F_n(z)| \le \gamma_n |z|^{-n-1} \qquad (*)$$

for all $|z| > \rho_n$ in S.

Property (B). For n = 0,1,2,... , we have

$$\lim_{\substack{z \to \infty \\ z \in S}} \left\{ z^n [f(z) - F_n(z)] \right\} = 0 .$$

Property (C). For n = 0,1,2,... , we have

$$\lim_{\substack{z \to \infty \\ z \in S}} \left\{ z^n [f(z) - F_{n-1}(z)] \right\} = a_n ,$$

where $F_{-1} = 0$.

Theorem(3.2.2). The properties (A) , (B) and (C) are equivalent.

Proof. It follows from (*) in property (A) that

$$| z^n [f(z) - F_n(z)] | \le \gamma_n |z|^{-1}$$

which goes to zero as $z \to \infty$. Hence (A) implies (B) .

Since $z^n F_n(z) = z^n(a_0 + a_1 z^{-1} + \dots + z^{-n+1}) + a_n = z^n F_{n-1}(z) + a_n$,

it follows that

$$z^n [f(z) - F_{n-1}(z)] = z^n [f(z) - F_n(z)] + a_n ,$$

and clearly (B) implies (C) .

Also, (C) implies that

$$a_{n+1} = \lim_{\substack{z \to \infty \\ z \in S}} \left\{ z^{n+1} [f(z) - F_n(z)] \right\} .$$

Thus, for $|z|$ sufficiently large,

$$|z^{n+1} [f(z) - F_n(z)]| \le 1 + |a_{n+1}|$$

$$|[f(z) - F_n(z)]| \le (1 + |a_{n+1}|) |z|^{-n-1} .$$

This is precisely (*) where $\gamma_n = 1 + |a_{n+1}|$.

Remark(3.2.3). All ρ_n may be assumed the same , because (*) in (A) for $n = 0$ implies that $|f(z) - a_0| \le \gamma_0 |z|^{-1}$ for $|z| > \rho_0$, and thus in particular f is bounded for $|z| > \rho_0$, $z \in S$. Hence for any n such that $\rho_n > \rho_0$, $z^{n+1}[f(z) - F_n(z)]$ is bounded, say by $\hat{\gamma}_n$, on the set S , $\rho_0 < |z| < \rho_n$. replacing γ_n by max $(\gamma_n , \hat{\gamma}_n)$, it follows that (*) holds for $|z| > \rho_0$, $z \in S$.

By virtue of Theorem (3.2.2) , properties (A) , (B) and (C) may be used as equivalent conditions for f to be represented asymptotically by F as $z \to \infty$ in S.

It is customary to use the notation

$$f \approx F, \quad as \ z \to \infty \ in \ S$$

to indicate that these conditions are satisfied.

If S is a full neighbourhood of infinity or the manner in which z is allowed to tend to infinity is unrestricted, then the qualifier " $z \in S$ " may be omited. Throughout we adopt the following notation

$$S_\alpha = \{ z : | Arg\, z | < \alpha \}$$

and $\overline{S}_\alpha$ is the closure of S_α .

Theorem(3.2.4). A function f can in a given unbounded set S has at most one APS as $z \to \infty$, $z \in S$.

Proof. Let $F = a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots$ *and* $G = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots$ be two distinct APS representations of f as $z \to \infty$ in S. Let n be the smallest integer such that $a_n \neq b_n$. Then $F_{n-1} = G_{n-1}$. Hence by (C)

$$a_n = \lim_{\substack{z \to \infty \\ z \in S}} \{ z^n [f(z) - F_{n-1}(z)] \} = \lim_{\substack{z \to \infty \\ z \in S}} \{ z^n [f(z) - G_{n-1}(z)] \} = b_n ,$$

which contradicts the definition of n.

Remark(3.2.5). The converse of this theorem is not true. An APS does not uniquly determine a function, for example if

$$f(z) \approx \sum_{k=0}^{\infty} a_k z^{-k} \quad as\ z \to \infty\ in\ \overline{S}_\alpha , \alpha < \frac{\pi}{2} ,$$

then also

$$f(z) + e^{-z} \approx \sum_{k=0}^{\infty} a_k z^{-k} \quad as\ z \to \infty\ in\ \overline{S}_\alpha , \alpha < \frac{\pi}{2} .$$

For the APS of f as z approach a finite value, we introduce the following definition :

Definition(3.2.6). Let f be defined on S and $z_0 \in \overline{S}$, and let

$$F = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

be a formal power series in $z - z_0$ with partial sums

$$F_n = \sum_{k=0}^{n} a_k (z - z_0)^k ,$$

$F_{-1} = 0$. Then F is called **asymptotic power series** , APS , of f as $z \to z_0$ in S if one of the properties (A),(B) and (C) is satisfied for z replaced by $(z - z_0)^{-1}$. In this case, we write

$$f \approx F \quad as \ z \to z_0 \ in \ S \ .$$

The analog of Theorem (3.2.4) holds also in this case, and the proof goes essentially in the same manner.

Theorem(3.2.7). A function f has at most one APS as $z \to z_0$ *in* S , $z_0 \in \overline{S}$.

Now we give some examples which illustrate the previous definitions. One can see that the main tool used is the following

$$\frac{1}{1-z} = 1 + z + z^2 + \dots + z^{n-1} + \frac{z^n}{1-z}, \quad \text{for } |z| < 1 \ .$$

We start by the following example :

Example(3.2.8). Consider the function

$$f(z) = \int_0^\infty \frac{h(\tau)}{z+\tau} \, d\tau$$

for $z \in S_\pi$, where h is a real continuous function on $[0,\infty)$, and such that all the integrals

$$c_k = \int_0^\infty \tau^k h(\tau) \ dt \ ,$$

$k = 0,1,2,\dots$, are absolutely convergent. We show that

$$f(z) \approx \sum_{k=0}^\infty (-1)^k c_k z^{-k-1} \ ,$$

as $z \to \infty$ in $\overline{S}_\alpha$, where $\alpha < \pi$. We have

$$\frac{1}{z+\tau} = \frac{1}{z}\left[1 - \tau z^{-1} + \tau^2 z^{-2} - \dots + (-1)^{n-1} \tau^{n-1} z^{-n+1} + (-1)^n \frac{(\tau/z)^n}{1+\tau/z} \right] ,$$

for $|z| > \tau$. Hence,

$$f(z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{z^{k+1}} \int_0^\infty \tau^k h(\tau)\, d\tau + \frac{(-1)^n}{z^n} \int_0^\infty \frac{\tau^n h(\tau)}{z+\tau}\, d\tau$$

$$= \sum_{k=0}^{n-1} (-1)^k c_k z^{-k-1} + R_n(z) .$$

where $R_n(z) = \frac{(-1)^n}{z^n} \int_0^\infty \frac{\tau^n h(\tau)}{z+\tau}\, d\tau$.

But, when $\operatorname{Re} z \geq 0$, $|z+\tau| \geq |z|$ for all $\tau \geq 0$, and when $\operatorname{Re} z < 0$, $|z+\tau| \geq |z| \sin\alpha$, $\alpha < \pi$. Using these estimates and the fact that

$$c_k = \int_0^\infty \tau^k h(\tau)\, dt ,$$

converges absolutely for $k = 0,1,2,\ldots$, we show that $R_n(z) = O(z^{-n-1})$, *as* $z \to \infty$ *in* $\overline{S}_\alpha$, $\alpha < \pi$. Therefore, the function f is asymptotic to the given series in the given domain.

A typical example of the function h is the exponential function e^{-t} . We may use this example to get the APS of other functions in integral form, as shown in the following example.

Example(3.2.9). Consider the function

$$f(z) = e^z \int_z^\infty \frac{e^{-t}}{t}\, dt ,$$

where $z \in S_\pi$. Letting $t = z + \tau$, we have

$$f(z) = \int_0^\infty \frac{e^{-t}}{z+\tau}\, dt .$$

By Example (3.2.8)

$$f(z) \approx \sum_{k=0}^\infty (-1)^k k!\, z^{-k-1} ,$$

as $z \to \infty$ in $\overline{S}_\alpha$, where $\alpha < \pi$. We notice here that the APS is divergent .

Example(3.2.10). (Binet Function)

In this example we find the APS of the Binet function , $J(z)$, defined by

$$J(z) = \frac{1}{\pi}\int_0^\infty \frac{z}{z^2+\eta^2} Log\,(1-e^{-2\pi\eta})^{-1} d\eta \ .$$

For $|z| > |\eta|$ we have

$$\frac{z}{z^2+\eta^2} = \sum_{m=0}^{k-1} (-1)^m (\eta^2)^m z^{-(2m+1)} + r_k \ ,$$

where $r_k = \dfrac{(-1)^k \eta^{2k}}{z^{2k-1}(z^2+\eta^2)}$. Then,

$$J(z) = \frac{1}{\pi}\sum_{m=0}^{k-1} (-1)^m z^{-2m-1} \int_0^\infty \eta^{2m} Log\,(1-e^{-2\pi\eta})^{-1} d\eta + \frac{1}{\pi}\int_0^\infty r_k\, Log\,(1-e^{-2\pi\eta})^{-1} d\eta \ .$$

Define $\beta_m = \frac{1}{\pi}\int_0^\infty \eta^{2m} Log\,(1-e^{-2\pi\eta})^{-1} d\eta$, for m = 0,1,2,... . Then, β_m exists and $\beta_m > 0$ for every m. Then

$$J(z) = \sum_{m=0}^{k-1} (-1)^m \beta_m z^{-2m-1} + R_k(z) \ ,$$

where

$$R_k(z) = \frac{(-1)^k}{\pi} z^{-2k+1} \int_0^\infty \frac{\eta^{2k}}{z^2+\eta^2} Log\,(1-e^{-2\pi\eta})^{-1} d\eta \ .$$

Following the same argument used to estimate the remainder in Example (3.2.6), we have

$$R_k(z) = O(z^{-(2k+1)}), \qquad as\ z\to\infty\ in\ \overline{S}_\alpha\ ,\ \alpha < \frac{\pi}{2} \ .$$

Therefore,

$$J(z) \approx \sum_{m=0}^{\infty} (-1)^m \beta_m z^{-}(2m+1) \qquad as\ z\to\infty\ in\ \overline{S}_\alpha\ ,\ \alpha < \frac{\pi}{2} \ .$$

Now we find the APS of a function defined by a series.

Example(3.2.11). Consider the function

$$f(z) = \sum_{k=1}^{\infty} \frac{\gamma^k}{z+k}$$

for $z \in S_\pi$ and $0 < \gamma < 1$. We show that for every $\alpha < \pi$,

$$f(z) \approx a_1 z^{-1} - a_2 z^{-2} + \dots \,, \quad as \ z \to \infty, \ in \ \overline{S}_\alpha \,,$$

where $a_n = \sum_{k=1}^{\infty} k^{n-1} \gamma^k$, n = 1,2,.... For $|z| > |k|$ we have

$$\frac{1}{z+k} = \frac{1}{z}\,\frac{1}{1+k/z} = \frac{1}{z}\sum_{j=0}^{n-1}(-1)^j\left(\frac{k}{z}\right)^j + S_{nk} \,,$$

where $S_{nk} = \dfrac{(-1)^n k^n}{z^n(z+k)}$. Hence,

$$f(z) = \sum_{k=1}^{\infty}\frac{\gamma^k}{z+k} = \sum_{k=1}^{\infty}\frac{\gamma^k}{z}\sum_{j=0}^{n-1}(-1)^j\left(\frac{k}{z}\right)^j + \sum_{k=1}^{\infty}\gamma^k S_{nk}$$

$$= \sum_{j=1}^{n}(-1)^j\left[\sum_{k=1}^{\infty}\gamma^k k^{j-1}\right] z^{-j} + \sum_{k=1}^{\infty}\gamma^k S_{nk}$$

$$= \sum_{j=1}^{n}(-1)^j a_j z^{-j} + R_n(z)$$

where $a_j = \sum_{k=1}^{\infty}\gamma^k k^{j-1}$ and $R_n(z) = \sum_{k=1}^{\infty}\gamma^k S_{nk}$. Substituting the value of S_{nk}, we have

$$|R_n(z)| \le \left(\sum_{k=1}^{\infty}\frac{\gamma^k k^n}{|z+k|}\right)|z|^{-n} \,.$$

But $|z+k| \ge |z|$ for $\mathrm{Re}\, z \ge 0$ or $|z+k| \ge |z|\sin\alpha$ for $\mathrm{Re}\, z < 0$. Moreover, the series $\sum_{k=1}^{\infty}\gamma^k k^n$ converges for $0 < \gamma < 1$. Hence $R_n(z) = O(z^{-n-1})$, $as \ z \to \infty$ $in \ \overline{S}_\alpha$, $\alpha < \pi$.

We end up this section by the following example.

Example(3.2.12). We find the APS for the exponential function

$$f(z) = e^{g(z)} ,$$

where g is a conformal mapping. Clearly, if g maps the domain S into the left-half plane with $g(z_0) = \infty$, z_0 can be infinity, then we have

$$f(z) \approx 0, \quad as\ z \to z_0\ in\ S .$$

Hence for different g we have

$$e^z \approx 0, \quad as\ z \to \infty\ in\ \overline{S}_\alpha,\ where\ \frac{\pi}{2} < \alpha < \frac{3\pi}{2} .$$

$$e^{-z} \approx 0, \quad as\ z \to \infty\ in\ \overline{S}_\alpha,\ where\ \alpha < \frac{\pi}{2}$$

$$e^{-z^2} \approx 0, \quad as\ z \to \infty\ in\ \overline{S}_\alpha,\ where\ \alpha < \frac{\pi}{4}$$

$$e^{-z^{-2}} \approx 0, \quad as\ z \to 0\ in\ \overline{S}_\alpha,\ where\ \alpha < \frac{\pi}{4}$$

and so on.

3.3 Operations on Power Series

We consider in this section four operations on APS : Addition, Multiplication, Reciprocal and Composition as follows .

Let

$$F = a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots$$

$$G = b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots$$

be two formal power series. We define the following operations :

Addition

$$\alpha F + \beta G = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) z^{-n}$$

for any complex numbers α and β .

Multiplication

$$F\,G = \sum_{n=0}^{\infty} c_n z^{-n} ,$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} b_k a_{n-k}$, $n = 0,1,2,\ldots$, which is the **Cauchy product** of the two series.

Reciprocal

If $a_0 \neq 0$, $\frac{1}{F}$ is defined to be the solution of the equation $F X = 1$.

The following theorems show that asymptotic power series are preserved under these operations.

Theorem(3.3.1). Let S be an unbounded set in the complex plane, the functions f and g be defined on S, and F and G be two formal power series such that

$$f(z) \approx F \quad and \quad g(z) \approx G \ , \ z\to\infty \ in \ S \ .$$

Then for any complex numbers α and β

(i) $(\alpha f + \beta g)(z) \approx \alpha F + \beta G$

(ii) $(f g)(z) \approx F\,G$

(iii) $\frac{1}{f(z)} \approx \frac{1}{F}$, if $a_0 \neq 0$

as $z\to\infty$ in S .

Proof. (i) By definition $(\alpha F + \beta G)_n = \alpha F_n + \beta G_n$, where the subscript denotes the partial sum. Using Property (A),

$$\begin{aligned} (\alpha f + \beta g)(z) - (\alpha F + \beta G)_n(z) &= \alpha\, [f(z) - F_n(z)\,] + \beta\, [g(z) - G_n(z)\,] \\ &= O(z^{-n-1}) + O(z^{-n-1}) \\ &= O(z^{-n-1}), \end{aligned}$$

for $n = 0,1,2,\ldots$. Thus the series $(\alpha F + \beta G)$ satisfies Property (A). This proves (i).

(ii) Although it is not true in general that $(FG)_n = F_n G_n$, we have $(FG)_n = F_n G_n + O(z^{-n-1})$. Hence

$$\begin{aligned} f(z)\,g(z) - (FG)_n(z) &= f(z)\,g(z) - F_n(z)\ G_n(z) + O(z^{-n-1}) \\ &= [f(z) - F_n(z)]\ g(z) + [g(z) - G_n(z)]\ F_n(z) + O(z^{-n-1}) \\ &= O(z^{-n-1})\,g(z) + O(z^{-n-1})\ F_n(z) + O(z^{-n-1}). \end{aligned}$$

Since the limits of $g(z)$ and $F_n(z)$ exists as $z \to \infty$ in S, the right-hand side is $O(z^{-n-1})$. This proves (ii).

(iii) From Property (C), $\lim\limits_{\substack{z\to\infty \\ z\in S}} f(z) = a_0$, $a_0 \neq 0$. Therefore, $\frac{1}{f}$ is defined for $|z|$ sufficiently large, $z \in S$. Let

$$G = \frac{1}{F} = \frac{1}{a_0} - \frac{a_1}{a_0^2} z^{-1} + \dots .$$

Then

$$\begin{aligned} \frac{1}{f(z)} - G_n(z) &= \frac{1 - f(z)\ G_n(z)}{f(z)} \\ &= \frac{1}{f(z)} \left\{ 1 - [\, F_n(z) + O(z^{-n-1})\,]\ G_n(z) \right\}. \end{aligned}$$

By the definition of G,

$$F_n(z)\ G_n(z) = 1 + O(z^{-n-1})$$

so we have

$$\frac{1}{f(z)} - G_n(z) = \frac{1}{f(z)} \left\{ O(z^{-n-1}) + G_n(z)\ O(z^{-n-1}) \right\} = O(z^{-n-1})$$

as $z \to \infty$ *in* S. Thus the series G satisfies Property (A) with regard to $\frac{1}{f}$. This proves (iii).

As an application of the above theorem we have the following example.

Example(3.3.2). Consider the function

$$f(z) = e^{-1/z^2} + \sin z$$

in a neighbourhood of the origin which is an essential singularity. From Example (3.2.12)

$$e^{-1/z^2} \approx 0 \ \textit{as} \ z \to 0 \ \textit{in} \ \overline{S}_\alpha \ \textit{where} \ \alpha < \frac{\pi}{4} \ .$$

Also,

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{(2k+1)}}{(2k+1)!} \ .$$

Hence

$$f(z) \approx \sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{(2k+1)}}{(2k+1)!} \ .$$

as $z \to 0$ *in* $\overline{S}_\alpha$, *where* $\alpha < \frac{\pi}{4}$. If we multiply f by e^{1/z^2} we get

$$e^{1/z^2} f(z) = 1 + e^{1/z^2} \sin z \ .$$

But $e^{1/z^2} \approx 0$ *as* $z \to 0$ *in* $\overline{S}_\alpha$, $\frac{\pi}{4} < \alpha < \frac{3\pi}{4}$. Hence,

$$e^{1/z^2} f(z) \approx 1 \ \textit{as} \ z \to 0 \ \textit{in} \ \overline{S}_\alpha \ , \ \frac{\pi}{4} < \alpha < \frac{3\pi}{4} \ .$$

Composition of Series

Let $F = a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots$, then the kth power of F , F^k , is again a power series in $\{z^{-n}\}$. We write

$$F^k = a_0^{(k)} + a_1^{(k)} z^{-1} + a_2^{(k)} z^{-2} + \dots .$$

If $G = b_0 + b_1 w + b_2 w^2 + \dots$, then the composition of G with F , denoted by $G \circ F$, is obtained by substituting F into G . That is

$$G \circ F = b_0 + b_1 F + b_2 F^2 + \dots ,$$

and if

$$G \circ F = c_0 + c_1 z^{-1} + c_2 z^{-2} + \dots ,$$

then

$$c_0 = b_0$$

$$c_n = b_1 a_n^{(1)} + b_2 a_n^{(2)} + \dots + b_n a_n^{(n)} ,$$

for $n = 1,2,\dots$.

The following theorem shows that the composition of functions possessing asymptotic power series is represented asymptotically by the composition of their asymptotic series.

Theorem(3.3.3). Let S be an unbounded set in the complex plane, f be defined on S and

$$f(z) \approx F = a_1 z^{-1} + a_2 z^{-2} + \dots \quad as \ z \to \infty \ in \ S$$

Let g be defined on $T = f(S)$ and

$$g(w) \approx G = b_0 + b_1 w + b_2 w^2 + \dots \quad as \ w \to 0 \ in \ T$$

then

$$(g \circ f)(z) \approx G \circ F \quad as \ z \to \infty \ in \ S .$$

Proof. By the previous Theorem (3.3.1) (ii),

$$[f(z)]^k \approx F^k(z)$$

or

$$[f(z)]^k = (F^k(z))_n + O(z^{-n-1})$$

Using the argument in the proof of (ii) in Theorem (3.3.1) inductively, we get

$$(F^k(z))_n = [F_n(z)]^k + O(z^{-n-1})$$

or

$$[f(z)]^k = [F_n(z)]^k + O(z^{-n-1})$$

Also,

$$\begin{aligned}(G_n\circ f)(z) &= b_0 + b_1 f(z) + \dots + b_n f^n(z)\\ &= b_0 + b_1\{\, F_n(z) + O(z^{-n-1})\,\} + b_2\{\,[F_n(z)]^2 + O(z^{-n-1})\,\}\\ &\quad + \dots + b_n\{\,[F_n(z)]^n + O(z^{-n-1})\,\}\\ &= (G_n\circ F_n)(z) + O(z^{-n-1}).\end{aligned}$$

But

$$(g\circ f)(z) = (G_n\circ f)(z) + O(f^{n+1}(z)) .$$

So by the above and that $f(z) = O(z^{-1})$ the proof is completed.

As an application of this theorem, we have for $\alpha < \pi$, $e^{-z} \approx 0$ *as* $z\to\infty$ *in* $\overline{S}_\alpha$. Also,

$$\log(1+w) \approx 0 \;\; as \;\; w\to 0 \;\; in \;\; \overline{S}_\alpha .$$

Hence,

$$\log(1-e^{-z}) \approx 0 \;\; as \;\; z\to 0 \;\; in \;\; \overline{S}_\alpha .$$

Theorem (3.3.3) is a special case of the following theorem, which can be proven in the similar way.

Theorem(3.3.4). Let S be a set in the complex plane such that $z_o \in S$, f be defined on S, and

$$f(z) \approx. F(z) \;\; as \;\; z\to z_o \;\; in \;\; S .$$

Let g be defined in $T = f(S)$, and

$$g(w) \approx G(w) \;\; as \;\; w\to w_o = f(z_o) \;\; in \;\; T .$$

Then

$$(g\circ f)(z) \approx (G\circ F)(z) \;\; as \;\; z\to z_o \;\; in \;\; S .$$

3.4 APS of Analytic Functions

Theorem(3.4.1). Let f be analytic function for $\rho < |z| < \infty$, $\rho > 0$. Then f has an APS for $z \to \infty$, where the approach is unrestricted, if and only if the singularity at ∞ is removable, in which case the APS is identical with the Laurent series of f at ∞ , and hence converges at least for $|z| > \rho$.

Proof.

(a) Suppose f is analytic and has a removable singularity at ∞ . Then f has a Laurent series of the form

$$\sum_{k=0}^{\infty} c_k z^{-k} .$$

Let $F_n(z) = c_0 + c_1 z^{-1} + \ldots + c_n z^{-n}$, $n = 0,1,2,\ldots$. Then for every $n = 0,1,2,\ldots$,

$$f(z) - F_n(z) = c_{n+1} z^{-n-1} + \ldots = O(z^{-n-1}) .$$

Hence, by Property (A), $f \approx F$ *as* $z \to \infty$, where $F = c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots$.

(b) Now, suppose f has the APS $F = c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots$ as $z \to \infty$, where the approach is unrestricted, then by (C) $\lim\limits_{z \to \infty} f(z) = c_0$, whatever the way of approach, exists. Hence infinity is a removable singularity of f . The Laurent series thus involves no positive powers of z, and by (a) also is an asymptotic series. By the uniqueness of APS, F must agree with the Laurent series, and thus is convergent.

According to this theorem we have two remarks :

Remarks(3.4.2).

(1) The previous theorem implies the following : If f is analytic for $\rho < |z| < \infty$ and has an APS for $z \to \infty$ that is not identical with its Laurent series, then infinity is an

essential singularity. For if it is a removable singularity then the APS coincides with the Laurent's series, and if it is a pole then $\lim_{z\to\infty} f(z) = \infty$ and in either case we have a contradiction.

(2) A function with an essential singularity at infinity may have an APS in $\{z^{-n}\}$ (which may be convergent) if the the approach to infinity is suitable restricted. As an example of this

$$e^{-z} \approx 0 \quad as \quad z\to\infty \ in \ \overline{S}_{\alpha}, \ \alpha < \frac{\pi}{2} .$$

3.5 *Asymptotic Sequences and Developments*

So far we have considered the behavior of a function, say in the neighbourhood of the origin or infinity, purely in terms of power series or inverse power series. that is we have used the monomial z^p, p is an integer, to " gauge " the behavior of the function. However, for many purposes this is too crude and to get a more accurate description of the asymptotic behavior one should use additional gauge functions, like e^z, z^λ (λ a complex constant), $\ln z$, and so on. We introduce arbitrary gauge functions through the definition of asymptotic sequence.

Definition(3.5.1). A sequence of functions $\{\varphi_n\}$, all defined in a common domain S , is said to be an **asymptotic sequence, AS** , as $z\to z_o \in S$ in S if

$$\varphi_{n+1} = o(\varphi_n) \quad as \quad z\to z_o \ in \ S .$$

Accordingly , $\{z^n\}$ and $\{z^{-n}\}$ for positive n are AS at the origin and infinity, respectively. Other examples of AS are :

$$\{e^{-nz}\} \quad as \quad z\to\infty \ in \ S_{\pi/2}$$

$$\{(\ln z)^{-n}\} \quad as \quad z\to\infty \ in \ S_{\pi}$$

in the following example we consider an important AS, which will be used later.

Example(3.5.2). Consider the sequence $\{z^{-\lambda_n}\}$, where $\{\lambda_n\}$ is a sequence of complex constants such that $\operatorname{Re}\lambda_{n+1} > \operatorname{Re}\lambda_n$. Letting $z = re^{i\theta}$ and $\lambda_n = x_n + y_n i$, we have for $0 \le \theta < \pi$

$$\left|\frac{z^{-\lambda_{n+1}}}{z^{-\lambda_n}}\right| = \left|\frac{z^{\lambda_n}}{z^{\lambda_{n+1}}}\right| = \left|\frac{e^{\lambda_n \ln z}}{e^{\lambda_{n+1}\ln z}}\right| = e^{(x_n - x_{n+1})\ln r - \theta(y_n - y_{n+1})} ,$$

which goes to zero as $|z|\to\infty$. Hence, the sequence $\{z^{-\lambda_n}\}$ is an AS in $\overline{S}_\alpha$, $\alpha < \pi$.

Having introduced general gauge functions by means of AS we can also generalize the definition of asymptotic series.

Definition(3.5.3). Let $f(z)$, $\{\varphi_n(z)\}$ be defined in S and $z_o \in \overline{S}$. , and $\{\varphi_n\}$ be an AS as $z\to z_o$. Then $f(z)$ is said to have an asymptotic development , AD , in $\{\varphi_n\}$ as $z\to z_o$ if there exist constants c_k such that one of the following properties is satisfied.

Property I. For n = 0,1,2,... , we have

$$f(z) - \sum_{k=0}^{n} c_k\varphi_k(z) = O(\varphi_{n+1}) \quad as\ z\to z_o\ in\ S\ .$$

Property II. For n = 0,1,2,... , we have

$$\lim_{z\to z_o, z\in S}\left\{\frac{f(z) - \sum_{k=0}^{n} c_k\varphi_k(z)}{\varphi_n(z)}\right\} = 0\ .$$

Property III. For n = 0,1,2,... , we have

$$\lim_{z\to z_o, z\in S}\left\{\frac{f(z) - \sum_{k=0}^{n-1} c_k\varphi_k(z)}{\varphi_n(z)}\right\} = c_n\ .$$

Remark(3.5.4). The above construction along with the above definition constitute the idea of asymptotic development in the sense of Poincare′ . The definition of AD has a more general form. The interested reader can see Erdelyi, A. [6] . But, for our purpose the given definition is enough.

We finish this section here by simply saying that most of the theory established for APS works for AD.

Chapter IV

ASYMPTOTIC BEHAVIOR OF LSI

In this chapter we study the relation between the asymptotic behavior of the determining function and the integral

$$f(s) = \int_0^\infty e^{-st}\, d\alpha(t)$$

4.1 *Order of the Determining Function and Convergence*

In this section , certain relations between the order of the determining function and the convergence of the corresponding LSI are established. Throughout this chapter we assume α to be of bounded variation in every interval $[0,R]$, $R > 0$.

Theorem(4.1.1). If $\alpha(t) = O(e^{\gamma t})$ as $t \to \infty$ for some real number γ , then the integral $f(s)$ converges for every s such that $\operatorname{Re} s > \gamma$.

Proof. Applying integration by parts,

$$\int_0^R e^{-st}\, d\alpha(t) = e^{-sR}\alpha(R) - \alpha(0) + s\int_0^R e^{-st}\alpha(t)\, dt \ .$$

From the hypothesis, there exists a constant M such that

$$|\alpha(t)| \le M e^{\gamma t} \quad as \quad t \to \infty \ .$$

Hence

$$\left|e^{-sR}\alpha(R)\right| \le M e^{(\gamma-\sigma)R} \quad as \quad R \to \infty \ ,$$

where $\sigma = \operatorname{Re} s$. So, if $\sigma > \gamma$, $e^{-sR}\alpha(R) \to 0$ as $R \to \infty$.

Also, for sufficiently large x *and* y

$$\left|\int_x^y e^{-st}\alpha(t)\,dt\right| \le M\int_x^y e^{-\sigma t}e^{\gamma t}\,dt = \frac{M}{\gamma-\sigma}e^{-(\sigma-\gamma)t}\Big|_x^y .$$

So if $\sigma > \gamma$, the integral $\int_0^\infty e^{-st}\alpha(t)\,dt$ converges. Hence

$$\int_0^\infty e^{-st}\,d\alpha(t) = s\int_0^\infty e^{-st}\alpha(t)\,dt - \alpha(0) , \quad \sigma > \gamma .$$

The converse of this theorem is not true since if $\alpha(t) = 1$ for all t then $f(s)$ converges for all s , yet $\alpha(t) \neq O(e^{\gamma t})$ for $\gamma < 0$.

Corollary(4.1.2). If $\alpha(\infty)$ exists and if

$$\alpha(t) - \alpha(\infty) = O(e^{\gamma t}), \ as\ t\to\infty ,$$

for some real number γ , then $f(s)$ converges for every s such that s such that $\mathrm{Re}\, s > \gamma$.

Proof. The proof follows directly from the previous theorem by using

$$\int_0^\infty e^{-st}\,d[\alpha(t) - \alpha(\infty)] = \int_0^\infty e^{-st}\,d\alpha(t) .$$

Note. The significance of the corollary appears in the case when $\alpha(t) \neq O(e^{\gamma t})$ for all $\gamma < 0$ but $\alpha(t) - \alpha(\infty) = O(e^{\gamma t})$ for some $\alpha < 0$.

Theorem(4.1.3). If $f(s)$ converges for some s with $\mathrm{Re}\, s = \gamma > 0$, then

$$\alpha(t) = o(e^{\gamma t}) \ as\ t\to\infty .$$

Proof. We have

$$\alpha(t) - \alpha(0) = \int_0^t d\,\alpha(u) = \int_0^t e^{su}e^{-su}\,d\alpha(u) = \int_0^t e^{su}\,d\beta(u)$$

where $\beta(t) = \int_0^t e^{-su}\,d\alpha(u)$. Integration by parts gives

$$\alpha(t) - \alpha(0) = e^{su}\beta(u)\Big|_0^t - s\int_0^t e^{su}\beta(u)\,du = e^{st}\beta(t) - s\int_0^t e^{su}\beta(u)\,du ,$$

or

$$[\,\alpha(t) - \alpha(0)\,]\,e^{-st} = \beta(t) - s\,e^{-st}\int_0^t e^{su}\beta(u)\,du .$$

For s with $\operatorname{Re} s = \gamma > 0$, $\beta(\infty)$ exists and

$$\lim_{t\to\infty}\{[\,\alpha(t) - \alpha(0)\,]\,e^{-st}\} = \beta(\infty) - \lim_{t\to\infty}\left\{s\,e^{-st}\int_0^t e^{su}\beta(u)\,du\right\}$$

$$= \lim_{t\to\infty}\left\{s\,e^{-st}\int_0^t e^{su}[\,\beta(\infty) - \beta(u)\,]\,du\right\} .$$

$$= 0 .$$

Hence $\alpha(t) - \alpha(0) = o(e^{\gamma t})$ as $t\to\infty$, which implies that

$$\alpha(t) = o(e^{\gamma t}) \ \ as \ \ t\to\infty .$$

Theorem(4.1.4). If $f(s)$ converges for some s with $\operatorname{Re} s = \gamma < 0$, then $\alpha(\infty)$ exists and

$$\alpha(t) - \alpha(\infty) = o(e^{\gamma t}) \ \ as \ \ t\to\infty .$$

Proof. Since $\gamma < 0$, $f(s)$ converges for $s = 0$ and $\alpha(\infty)$ exists. For s with $\operatorname{Re} s = \gamma$ we have

$$\alpha(\infty) - \alpha(t) = \int_t^\infty e^{su}\,d\beta(u) ,$$

where $\beta(t) = \int_0^t e^{-su}\,d\alpha(u)$. Integration by parts gives

$$\alpha(\infty) - \alpha(t) = -e^{st}\beta(t) - s\int_t^\infty e^{su}\beta(u)\,du .$$

Taking the limit of both sides, we get

$$\lim_{t\to\infty}\{[\,\alpha(\infty) - \alpha(t)\,]e^{-st}\} = -\beta(\infty) - \lim_{t\to\infty}\left\{s\,e^{-st}\int_t^\infty e^{su}\beta(u)\,du\right\} .$$

$$= \lim_{t\to\infty} \left\{ s\, e^{-st} \int_t^{\infty} e^{su} [\, \beta(\infty) - \beta(u) \,]\, du \right\} .$$

The limit on the right side is zero. This proves the theorem.

Remark(4.1.5). Theorems (4.1.3,4) are both false when $\gamma = 0$. In the case of Theorem (4.1.3), if $\alpha(0) = 0$, $\alpha(t) = 1$, $t > 0$, $f(s)$ converges for all s, yet $\alpha(t) \neq o(1)$ *as* $t \to \infty$.

In the case of Theorem (4.1.4), if

$$\alpha(t) = \begin{cases} 2 & \text{if } t \le 1 \\ 2\sqrt{t} & \text{if } t > 1 \end{cases}$$

then

$$\int_0^{\infty} e^{-st}\, d\alpha(t) = \int_1^{\infty} \frac{e^{-su}}{\sqrt{u}}\, du .$$

The second integral converges for $s = i$, but $\alpha(\infty)$ does not exist.

Using Theorem (4.1.3), we can express the LSI in terms of the Laplace integral as follows.

Theorem(4.1.6). If $f(s)$ converges with $\operatorname{Re} s > 0$, then

$$f(s) = s \int_0^{\infty} e^{-st}\, \alpha(t)\, dt - \alpha(0) .$$

Proof. By integration by parts,

$$\int_0^{R} e^{-st}\, d\alpha(t) = e^{-sR}\, \alpha(R) - \alpha(0) + s \int_0^{R} e^{-st}\, \alpha(t)\, dt .$$

By Theorem (4.1.3), $e^{-sR}\, \alpha(R) \to 0$ *as* $R \to \infty$, and the result is obtained.

Also using Theorem (4.1.4) we obtain

Theorem(4.1.7). If $f(s)$ converges for s with $\operatorname{Re} s < 0$, then $\alpha(\infty)$ exists and

$$f(s) = \alpha(\infty) - \alpha(0) + s\int_0^\infty e^{-st}[\,\alpha(t) - \alpha(\infty)\,]\,dt\ .$$

Proof. We can write

$$f(s) = \int_0^\infty e^{-st}\,d[\,\alpha(t) - \alpha(\infty)\,]\ .$$

By integration by parts

$$\int_0^R e^{-st}\,d[\,\alpha(t) - \alpha(\infty)\,] = e^{-sR}[\,\alpha(R) - \alpha(\infty)\,] - [\,\alpha(0) - \alpha(\infty)\,] + s\int_0^R e^{-st}[\,\alpha(0) - \alpha(\infty)\,]\,dt\,.$$

By Theorem (4.1.4), $e^{-sR}[\,\alpha(R) - \alpha(\infty)\,] \to 0$ as $R \to \infty$, and the result is obtained.

4.2 LSI at the Origin and Infinity

In this section we show that the value of

$$f(s) = \int_0^\infty e^{-st}\,d\alpha(t)$$

at the origin (infinity) is equal to the value of the determining function , α , at infinity (the origin).

Theorem(4.2.1). If $f(s)$ is defined for $s > 0$ and $\alpha(\infty)$ exists, then $f(0^+)$ exists and

$$f(0^+) = \alpha(\infty) - \alpha(0)\ .$$

Proof. By Theorem (4.1.6), for $s > 0$ we have

$$f(s) = s\int_0^\infty e^{-st}\,\alpha(t)\,dt - \alpha(0)\ .$$

Let $x = st$, then

$$f(s) = \int_0^\infty e^{-x}\,\alpha(x/s)\,dx - \alpha(0)$$

and

$$\lim_{s \to 0^+} \{ f(s) - \alpha(\infty) \} = \lim_{s \to 0^+} \left\{ \int_0^\infty e^{-x} \alpha(x/s)\, dx - \alpha(\infty) \right\} - \alpha(0)$$

$$= \lim_{s \to 0^+} \left\{ \int_0^\infty e^{-x} [\, \alpha(x/s) - \alpha(\infty) \,]\, dx \right\} - \alpha(0)$$

$$= -\alpha(0)$$

since $x > 0$. This proves the theorem.

As a corollary we have

Corollary(4.2.2). If $f(s)$ is defined for $s = 0$ then

$$f(0) = \alpha(\infty) - \alpha(0) .$$

Proof. Since $f(s)$ converges for $s = 0$, $\alpha(\infty)$ exists . By Theorem (4.2.1) and the continuity of f the result is obtained.

Similarly, we can prove

Theorem(4.2.3). If $f(s)$ is defined for some s , then $f(\infty)$ exists and

$$f(\infty) = \alpha(0^+) - \alpha(0)$$

Proof. From the hypothesis, $f(s)$ is defined for some $s_0 > 0$. Hence, by Theorem (4.1.6), for $s > s_0$ we have

$$f(s) = s \int_0^\infty e^{-st} \alpha(t)\, dt - \alpha(0) = \int_0^\infty e^{-x} \alpha(x/s)\, dx - \alpha(0)$$

where $x = st$. Taking the limit we have

$$\lim_{s \to \infty} \{ f(s) - \alpha(0^+) \} = \lim_{s \to \infty} \int_0^\infty e^{-x} [\, \alpha(x/s) - \alpha(0^+) \,]\, dx - \alpha(0) = -\alpha(0) .$$

This completes the proof.

Corollary(4.2.4). If $g \in R$ and

$$G(s) = \int_0^\infty e^{-st} g(t)\, dt$$

is defined for some s, then

$$\lim_{s\to\infty} G(s) = 0$$

Proof. Let $\alpha(t) = \int_0^t g(t)\, dt$. Then $G(s) = \int_0^\infty e^{-st}\, d\alpha(t)$. By Theorem (4.2.3)

$$\lim_{s\to\infty} G(s) = \alpha(0^+) - \alpha(0) = 0 ,$$

since α is absolutely continuous in $[0,\infty)$.

4.3 Watson's Lemma for the LSI

Whereas Theorems (4.2.1) and (4.2.3) give information on the existence of $f(0^+)$ and $f(\infty)$ and thier values in terms of $\alpha(0)$, $\alpha(0^+)$ and $\alpha(\infty)$, one would be interested in capturing the asymptotic behavior of f at ∞ . It is the purpose of this section to show that the asymptotic behavior as $s\to\infty$ in some sector $\overline{S}_\beta$ of $f(s)$ is fully determined by the asymptotic behavior of the determining function α near the origin. Consequently, we show that if α has an asymptotic expansion in $\{t^{\gamma_n}\}$ near the origin then we can obtain an asymptotic expansion of $f(s)$ in $\{s^{\gamma_n}\}$ near infinity, whose coefficients are easily calculated from the asymptotic series of α .

First we prove the following lemma which we may call it Watson's Lemma for the LSI.

Theorem(4.3.1). Watson's Lemma for the LSI

If $f(s)$ converges for some s , $\int_0^\infty |\alpha(t)|\, dt < \infty$ and $\alpha(t) = O(t^\gamma)$ as $t\to 0^+$, for some γ where $\mathrm{Re}\,\gamma > -1$, then

$$f(s) + \alpha(0) = O(s^{-\gamma})$$

as $s \to \infty$ *in* $\overline{S}_\beta$, $\beta < \frac{\pi}{2}$.

Proof. From the hypothesis, $f(s)$ converges for some s with $\operatorname{Re} s = \sigma > 0$, and by Theorem (4.1.6)

$$f(s) + \alpha(0) = s \int_0^\infty e^{-st} \alpha(t)\, dt, \quad \operatorname{Re} s > 0.$$

Also, from the hypothesis

$$|\alpha(t)| < A\, t^\gamma \ \ as \ \ t \to 0^+$$

for some constant A. So for sufficiently small a we have

$$|f(s) + \alpha(0)| = \left| s \int_0^a e^{-st} \alpha(t)\, dt + s \int_a^\infty e^{-st} \alpha(t)\, dt \right|$$

$$\leq A\,|s| \int_0^a e^{-\sigma t} t^\gamma\, dt + |s|\, e^{-\sigma a} \int_a^\infty |\alpha(t)|\, dt$$

Clearly, the second integral vanishes as $s \to \infty$ in $\overline{S}_\beta$, $\beta < \frac{\pi}{2}$. To find the first integral let $x = \sigma t$, then

$$\int_0^a e^{-\sigma t} t^\gamma\, dt = \int_0^{\sigma a} e^{-x} (x/\sigma)^\gamma \frac{dx}{\sigma} = \frac{1}{\sigma^{\gamma+1}} \int_0^{\sigma a} e^{-x} x^\gamma\, dx \leq \frac{1}{\sigma^{\gamma+1}} \int_0^\infty e^{-x} x^\gamma\, dx = \frac{1}{\sigma^{\gamma+1}} \Gamma(\gamma+1).$$

Observing that $\sigma = |s| \cos\theta$, where $\theta = Arg\, s$, gives us the result.

Now, we introduce the following relations which will be used in the coming part. For any complex number z we define

$$(z)_n = z\,(z+1)\,(z+2)\ldots(z+n-1)$$

and for $\operatorname{Re} z > -1$

$$z! = \Gamma(z+1)$$

where Γ is the Gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \ ,$$

where $\operatorname{Re} z > 0$. Using these , we can evaluate the commonly used integral $\int_0^\infty e^{-st}\, t^z\, dt$.

Letting $\tau = st$,

$$\int_0^\infty e^{-st}\, t^z\, dt = \frac{1}{s}\int_0^\infty e^{-\tau}\, (\tau/s)^z\, d\tau = \frac{1}{s^{z+1}}\int_0^\infty e^{-\tau}\, \tau^z\, d\tau = \frac{z!}{s^{z+1}} \ .$$

Having these relations , we prove the following theorem which provides us with an asymptotic expansion of $f(s)$ known the asymptotic expansion of α .

Theorem(4.3.2). Suppose

$$\alpha(t) = \sum_{n=0}^{N} c_n t^{\gamma_n} + \rho_N(t)$$

in a neighbourhood of 0 such that $\operatorname{Re}\gamma_n = \tau_n$ and $-1 \le \tau_0 < \tau_1 < \dots < \tau_N$. If

(i) $\int_0^\infty |\rho_N(t)|\, dt < \infty$,

(ii) $\int_0^\infty e^{-st}\, d\rho_N(t)$ exists for some s and

(iii) $\rho_N(t) = O(t^{\gamma_{N+1}})$, $\operatorname{Re}\gamma_{N+1} = \tau_{N+1} > -1$, as $t \to 0^+$,

then

$$f(s) = \sum_{n=0}^{N} c_n\, \gamma_n!\, s^{-\gamma_n} + r_N(s)$$

with $r_N(s) = O(s^{-\gamma_{N+1}})$ *as* $s \to \infty$ *in* $\overline{S}_\beta$, $\beta < \dfrac{\pi}{2}$.

Proof. The LSI is linear, so we have

$$\int_0^\infty e^{-st}\, d\alpha(t) = \sum_{n=0}^{N} c_n \int_0^\infty e^{-st}\, d(t^{\gamma_n}) + \int_0^\infty e^{-st}\, d\rho_N(t)$$

$$= \sum_{n=0}^{N} c_n\, \gamma_n \int_0^\infty e^{-st}\, t^{\gamma_n - 1}\, dt + r_N(s)$$

$$= \sum_{n=0}^{N} c_n \gamma_n ! \, s^{-\gamma_n} + r_N(s)$$

By Watson's Lemma ,

$$r_N(s) = O(s^{-\gamma_{N+1}}) - \rho_N(0)$$

as $s \to \infty$ in $\overline{S}_\beta$, $\beta < \frac{\pi}{2}$. From hypothesis, for sufficiently small t there exists $M > 0$ such that

$$| \rho_N(t) | \leq M \, | t^{\gamma_{N+1}} | = M \, t^{\tau_{N+1}} .$$

So if $\tau_{N+1} \geq 0$ then $\rho_N(0)$ must be zero and $r_N(s) = O(s^{-\gamma_{N+1}})$. If $\tau_{N+1} < 0$ then

$$O(s^{-\gamma_{N+1}}) - \rho_N(0) = O(s^{-\gamma_{N+1}}) .$$

This shows that

$$r_N(s) = O(s^{-\gamma_{N+1}})$$

and completes the proof.

4.4 Watson's Lemma for the Laplace Integral

In this section we develop the analog of the theorems of the previous section for the Laplace Integral defined by

$$\varphi(z) = \int_0^\infty e^{-zt} \psi(t) \, dt$$

if the improper integral converges. Also, we get a generalized form of this lemma for analytic functions.

Theorem(4.4.1). If $\psi \in R$ and satisfies

(i) $\int_0^\infty | \psi(t) | \, dt < \infty$,

(ii) $\psi(t) = O(t^{\gamma})$, for $\operatorname{Re}\gamma > -1$ *as* $t \to 0^{+}$,

then

$$\varphi(z) = \int_0^{\infty} e^{-zt}\psi(t)\,dt = O(z^{-(\gamma+1)})$$

for $z \to \infty$ *in* $\overline{S}_{\beta}$, $\beta < \frac{\pi}{2}$.

Proof. From the hypothesis, $\varphi(z)$ converges for $\operatorname{Re} z = \sigma > 0$. Also, as $t \to 0^{+}$

$$|\psi(t)| < A\,t^{\gamma}$$

for some constant A . So for sufficiently small a we have

$$|\varphi(z)| \le A\int_0^{a} e^{-\sigma t}t^{\gamma}\,dt + e^{-\sigma a}\int_a^{\infty}|\psi(t)|\,dt\ .$$

As $z \to \infty$ in $\overline{S}_{\beta}$, $\beta < \frac{\pi}{2}$, the second integral vanishes and we have

$$|\varphi(z)| \le A\,\frac{\Gamma(\gamma+1)}{\sigma^{\gamma+1}} = A\,\Gamma(\gamma+1)\,\cos\theta\,|z|^{-\gamma-1}\ ,$$

where $\theta = \operatorname{Arg} z$. This completes the proof.

This theorem has a wide use in practical problems. As before, we use this theorem to derive an asymptotic expansion of the Laplace Integral , φ , of ψ at infinity from the asymptotic expansion of ψ at the origin.

Theorem(4.4.2). Suppose $\psi \in R$ and

$$\psi(t) = \sum_{n=0}^{N} c_n t^{\gamma_n} + \rho_N(t)$$

in a neighbourhood of 0 such that $\operatorname{Re}\gamma_n = \tau_n$ and $-1 \le \tau_0 < \tau_1 < \ldots < \tau_N$. If

(i) $\int_0^{\infty} |\rho_N(t)|\,dt < \infty$,

(ii) $\rho_N(t) = O(t^{\gamma_{N+1}})$ *as* $t \to 0^{+}$,

then

$$\varphi(z) = \frac{1}{z}\sum_{n=0}^{N} c_n\gamma_n!\, z^{-\gamma_n} + r_N(z)$$

with $r_N(z) = O(z^{-\gamma_{N+1}-1})$ *as* $s \to \infty$ *in* $\bar{S}_\beta$, $\beta < \frac{\pi}{2}$.

Proof. The proof follows directly from the linearity of the Laplace integral and the application of Theorem (4.4.1).

Remark(4.4.3). Theorem (4.3.1) still holds if we replace the condition $\int_0^\infty |\psi(t)| < \infty$ by the condition $\psi = O(e^{\delta t})$ as $t \to \infty$, for some real δ.

The restriction to $S_{\pi/2}$ in Watson's Lemma can be considerably relaxed if ψ is analytic in a secter including the real axis. To achieve this we need to show that φ assumes an analytic continuation beyond $S_{\pi/2}$ as shown in the following lemma.

Lemma(4.4.4). If ψ is analytic in $\bar{S}_\beta$ and $\psi(w) = O(e^{\gamma w})$ as $|w| \to \infty$ in $\bar{S}_\beta$, where γ is a complex constant, with $\operatorname{Re}\gamma > -1$, then for $\operatorname{Re} z > \operatorname{Re}\gamma$ the function

$$\varphi(z) = \int_0^\infty e^{-zt}\psi(t)\, dt$$

is analytic and has an analytic continuation defined by

$$\varphi_\theta(z) = \int_{P(\theta)} e^{-zw}\psi(w)\, dw ,$$

where $P(\theta) = \{re^{i\theta} : r > 0\}$, that determines a single-valued analytic function throughout the domain

$$\left\{ z : |\arg z - \theta| < \frac{\pi}{2},\ |\theta| < \beta \text{ and } |z| > |Re\,\gamma| \right\}.$$

Proof. It follows from the hypothesis that φ_θ is convergent and analytic in the half plane

$$D_\theta = \left\{ z : |\arg z - \theta| < \frac{\pi}{2} \text{ and } \operatorname{Re}(z e^{-i\theta} - \gamma) > 0 \right\}.$$

Now we show that the integrals φ_θ coincide in their common domains. That is, if $z \in D_{\theta_1} \cap D_{\theta_2}$, then $\varphi_{\theta_1}(z) = \varphi_{\theta_2}(z)$. For $R > 0$ and $|\theta| < \beta$, let

$$P(\theta, R) = \{t e^{i\theta} : t \text{ from } 0 \text{ to } R\}$$

$$C_R(\theta_2, \theta_1) = \{R e^{i\theta} : \theta \text{ from } \theta_2 \text{ to } \theta_1\}.$$

We consider temporarily the case when $\operatorname{Re}\gamma \geq 0$. Without loss of generality, we choose θ_1 and θ_2 such that $\theta_1 - \theta_2$ is sufficiently small so that $C_R(\theta_2, \theta_1)$ lies in $D_{\theta_1} \cap D_{\theta_2}$ for $R > \operatorname{Re}\gamma$. It follows by Cauchy's Theorem that

$$\int_{P(\theta_1,R)} e^{-zw}\psi(w)\,dw = \int_{P(\theta_2,R)} + \int_{C_R(\theta_2,\theta_1)} e^{-zw}\psi(w)\,dw .$$

But

$$\int_{C_R(\theta_2,\theta_1)} e^{-zw}\psi(w)\,dw \to 0 \quad as \quad R \to \infty .$$

Hence, as $R \to \infty$, $\varphi_{\theta_1} = \varphi_{\theta_2}$ on $D_{\theta_1} \cap D_{\theta_2}$. Therefore, we can define a single valued analytic function Φ_θ on $D_{\theta_1} \cup D_{\theta_2}$ as follows.

$$\Phi_\theta = \begin{cases} \varphi_{\theta_1} & on\ D_{\theta_1} \\ \varphi_{\theta_1} & on\ D_{\theta_1}. \end{cases}$$

However, if $\theta_1 = 0$ and $\theta_2 = \theta$ and θ is not sufficiently small so that $C_R(\theta, 0)$ does not lie in $D_0 \cap D_\theta$, we partition $[0,\theta]$ into small subdivisions $0 < \theta_1 < \theta_2 < \ldots < \theta_n = \theta$ such that $\theta_{i+1} - \theta_i$ is small enough so that $C_R(\theta_{i+1}, \theta_i)$ lies in $D_{\theta_{i+1}} \cap D_{\theta_i}$. Then, in view of the above argument we can construct a single valued analytic function Φ_β on $\bigcup_{|\theta|<\beta} D_\theta$ where for any two complex numbers $re^{i\alpha} = re^{i\gamma}$ if and only if $\alpha = \gamma$. Clearly Φ_β is the analytic continuation of φ_0.

Now we can prove the generalized form of Watson's Lemma.

Theorem (4.4.5). If in the sector $\overline{S}_\alpha$, $\alpha < \pi$

(i) $\psi(w)$ is analytic ,

(ii) $\psi(w) = O(e^{\gamma w})$ as $|w| \to \infty$, γ is a constant,

(iii) $\psi(w) = \sum_{n=0}^{N} c_n w^{\gamma_n} + \rho_N(w)$

with $\operatorname{Re} \gamma_n = \tau_n$ and $-1 < \tau_0 \le \tau_1 \le \ldots \le \tau_N$,

(iv) $\rho_N(w) = O(w^{\gamma_{N+1}})$, $\operatorname{Re} \gamma_{N+1} = \tau_{N+1} > -1$, as $|w| \to 0$ in S_α ,

then

$$\varphi(z) = \sum_{n=0}^{N} c_n \gamma_n! \, z^{-\gamma_n - 1} + r_N(z)$$

with $r_N(z) = O(z^{-\gamma_{N+1}-1})$ *as* $|z| \to \infty$ *in* $\overline{S}_\beta$, $\beta < \frac{\pi}{2} + \alpha$.

Proof. From Lemma (4.4.4), the analytic continuation of φ is given by

$$\varphi(z) = \int_{P(\theta)} e^{-zw} \psi(w) \, dw .$$

throughout $\{ z : | \arg z - \theta | < \pi/2 , \ |\theta| < \alpha , \text{ and } \operatorname{Re} z > \operatorname{Re} \gamma \}$. If we let $w = te^{i\theta}$, then

$$\psi(te^{i\theta}) = \sum_{n=0}^{N} c_n e^{i\gamma_n \theta} t^{\gamma_n} + \beta_N(t)$$

with $\beta_N(t) = O(t^{\gamma_{N+1}})$. Clearly, $\beta_N(t)$ is exponentially bounded as a function of t , and by Remark (4.4.3) and Theorem (4.4.1) the proof is completed.

4.5 *Applications on Watson's Lemma*

In this section we present some examples which demonstrate the power of Watson's Lemma, particularly in deriving asymptotic expansions of some functions in integral form.

Example(4.5.1). Consider for real x the function

$$f(x) = \int_0^\infty e^{-xt} \log(1+t^2)\, dt \; .$$

For $t^2 < 1$ we have

$$\log(1+t^2) = t^2 \left[1 - \frac{t^2}{2} + \ldots \right] .$$

Hence, by Theorem (4.4.2)

$$f(x) \approx \frac{2!}{x^3} - \frac{1}{2}\frac{4!}{x^5} + \ldots \qquad as \quad x \to \infty \; .$$

Example(4.5.2). The Incomplete Gamma Function

The Incomplete Gamma Function is defined by

$$\Gamma(a, z) = \int_z^\infty e^{-s} s^{a-1}\, ds \; ,$$

where $|\arg z| < \pi$, $|\arg a| < \pi$ and the path of integration runs parallel to the real axis. Letting $s = z(1+\tau)$, where $0 \le \tau < \infty$, we obtain

$$\Gamma(a, z) = z^a e^{-z} \int_0^\infty e^{-z\tau} (1+\tau)^{a-1}\, d\tau \; ,$$

where all powers having their principal values. The integral now is in the form considered in Theorem (4.4.5) where

$$\varphi(t) = (1+t)^{a-1}$$

and φ satisfies the conditions in $\overline{S}_\alpha$ for every $\alpha < \pi$. Because φ is analytic at $t = 0$, it is asymptotic to its Taylor series

$$\varphi(t) \approx \sum_{n=0}^{\infty} (-1)^n \frac{(1-a)_n}{n!} t^n \qquad as \quad t \to 0 \; .$$

Hence,

$$\Gamma(a, z) \approx z^{a-1} e^{-z} \left\{ 1 - \frac{(1-a)_1}{z} + \frac{(1-a)_2}{z^2} - \ldots \right\} ,$$

as $z \to \infty$, $z \in \overline{S}_\beta$, $\beta < \frac{3\pi}{2}$.

Example(4.5.3). The Complex Error Integral

Let

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\, ds ,$$

where the path of integration is the ray $\arg s = \arg z$. It is well known that

$$\int_0^\infty e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2} .$$

Hence, by Cauchy's Theorem

$$erf(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2}\, ds ,$$

where the path of integration is parallel to the real axis or, if $|\arg z| < \frac{\pi}{4}$, along the ray $\arg s = \arg z$. To reduce the integral to a form of Watson's Lemma, set $s^2 = z^2(1+\tau)$. This yields

$$erf(s) = 1 - \frac{z e^{-z^2}}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\tau z^2}}{\sqrt{1+\tau}}\, d\tau .$$

Again, the function

$$\varphi(t) = (1+\tau)^{-1/2} \approx \sum_{n=0}^\infty (-1)^n \frac{(1/2)_n}{n!} t^n , \quad t \to 0$$

satisfies the conditions of Theorem (4.4.5) in $\overline{S}_\alpha$, for every $\alpha < \pi$. Hence,

$$erf(s) \approx 1 - \frac{e^{-z^2}}{\sqrt{\pi}\, z} \left\{ 1 - \frac{(1/2)_1}{z^2} + \frac{(1/2)_2}{z^4} - \dots \right\} ,$$

as $z \to \infty$ *in* $\overline{S}_\beta$, $\beta < \frac{3\pi}{4}$.

Example(4.5.4). Product of Hankel Functions

Consider the product of Hankel Functions

$$H_\nu^{(1)}(z)\ H_\nu^{(2)}(z) \ = \ \frac{i}{\pi}\ e^{\nu\pi i/2} \int_0^\infty \frac{e^{-\tau z^2}}{\tau}\ e^{1/(2\tau)}\ H_\nu^{(1)}\left(\frac{i}{2\tau}\right) d\tau \ .$$

Let

$$\varphi(t) \ = \ \frac{i\, e^{i\nu\pi/2}}{\pi}\ \frac{e^{1/(2t)}}{t}\ H_\nu^{(1)}\left(\frac{i}{2t}\right) ,$$

then the integral will be

$$H_\nu^{(1)}(z)\ H_\nu^{(2)}(z) \ = \ \int_0^\infty e^{-\tau z^2}\ \varphi(\tau)\ d\tau \ ,$$

which similar to the form given in Theorem (4.4.5), where

$$\varphi(t) \ \approx \ \frac{2}{\pi^{3/2}\sqrt{t}} \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{1}{2}-\nu\right)_n \left(\frac{1}{2}+\nu\right)_n}{n!}\ t^n \ ,$$

as $t \to 0$, $t \in \overline{S}_\beta$, $\beta < \dfrac{3\pi}{2}$. Hence,

$$H_\nu^{(1)}(z)\ H_\nu^{(2)}(z) \ \approx \ \frac{2}{\pi} \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{1}{2}-\nu\right)_n \left(\frac{1}{2}+\nu\right)_n \left(\frac{1}{2}\right)_n}{n!}\ z^{-(2n+1)} \ ,$$

as $z \to \infty$, $z^2 \in \overline{S}_\beta$, $\beta < 2\pi$, that is, $z \in \overline{S}_\alpha$, $\alpha < \pi$.

It should be noted that the divergence of the series for φ caused the series of $\int_0^\infty e^{-zt}\,\varphi(t)\,dt$ to diverge more strongly than in the examples previously considered.

4.6 Numerical Observations

In this section, we show two numerical aspects. First, the optimal N such that the Nth partial sum gives the best approximation. Second, the priority of divergent asymptotic power series to slowly convergent series in approximation, from the numerical point of view. These are discussed in the following two examples.

Example I

Example (4.5.2) shows that for $a = 0$ and $z = x > 0$

$$\Gamma(0,x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \approx e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ldots\right],$$

as $x \to \infty$, with the remainder after n terms

$$r_{n+1}(x) = (-1)^n \frac{e^{-x}}{x^n} \int_0^\infty \frac{e^{-t} t^n}{x+t}\,dt .$$

The APS is divergent for any x since the nth term tends to infinity as $n \to \infty$. Of course, $r_{n+1}(x)$ is also unbounded as $n \to \infty$. But

$$| r_{n+1}(x) | \leq \frac{e^{-x} n!}{x^{n+1}} .$$

Thus if n is fixed, $r_n(x) \to 0$ as $x \to \infty$. Consequently, for large x, the summation of the first n terms gives a good approximation to $\Gamma(0,x)$ and the error is small.

The question now is how we choose the optimal n so that the nth partial sum gives a best approximation to $\Gamma(0,x)$ for a given x. Table I shows that for $x = 8.6$ the terms $\frac{n!}{x^{n+1}}$ at first decrease rapidly. However, terms with $n > 9$ start to increase.

This behavior is justifiable. The ratio of the (n+1)th term to the nth term in the series is equal to $n x^{-1}$, so for $n < x$ the ratio is less than 1 so the terms decrease. While for $n > x$ the ratio is greater than 1 so the terms increase. Hence the Nth term, where N is equal to the largest integral part of the given x, is the smallest term. Therefore the bound on the remainder is minimum if we sum up to the (N - 1)th term, since it is bounded by the first neglected term.

The practical procedure for calculating $\Gamma(0,x)$ for a given fixed x is to evaluate successive terms in the series and to terminate the series when we reach the term which is greater than the previous one.

X=8.6

n	nth term	summation
0	0.116279006	0.116279006
1	-0.135208108E-01	0.102758169
2	0.314437435E-02	0.105902493
3	-0.109687424E-02	0.104805589
4	0.510173617E-03	0.105315745
5	-0.296612503E-03	0.105019093
6	0.206938945E-03	0.105225980
7	-0.168438666E-03	0.105057538
8	0.156687107E-03	0.105214179
9	-0.163974866E-03	0.105050206
10	0.190668405E-03	0.105240822
11	-0.243878167E-03	0.104996920
12	0.340295024E-03	0.105337203
13	-0.514399260E-03	0.104822755
14	0.837394036E-03	0.105660141
15	-0.146057038E-02	0.104199529

Table I

The series in this example is a typical example of a class of series which used to be called **semi-convergent** or **Stieltjes series**, for which the terms first decrease then increase. Such series are widely used to approximate functions numerically since computers deals with only finite number of terms, and for which convergently beginning series is efficient.

In addition, the series has the property that the error is bounded by the first neglected term. Consequently, for such series, for a given value of x, there exists a best approximation but only a definite accuracy can be achieved. These two properties are satisfied by most asymptotic expansions occur in practice.

Example II

Convergent series are not necessarily of practical value from a computational point of view unless it is rapidly convergent in which case only a few terms are needed. Asymptotic approximation may otherwise be chosen which gives divergent series that sometimes are of more practical use than the convergent series. Such a divergent series may surprisingly be remarkably accurate.

The following example demonstrates some of the differences between convergent and asymptotic series. Consider the function defined by

$$\gamma(a,x) = \int_0^x e^{-t}\, t^{a-1}\, dt \ ,$$

where x and a are positive. For $t > 0$ we have

$$e^{-t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, t^n$$

where the series is uniformly convergent in $[\,\delta,\infty]$ for any $\delta > 0$. By integrating term by term we get

$$\gamma(a,x) = x^a \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(a+n)}\, x^n$$

The series is convergent for all x , so it seems that it can be used to calculate $\gamma(a,x)$ for all x . Unfortunately, for large values of x the series is not a good approximation since a large number of terms is needed to get a small error. Table II demonstrates this fact. However, for large x we can obtain a different asymptotic series by writing

$$\begin{aligned} \gamma(a,x) &= \int_0^{\infty} e^{-t}\, t^{a-1}\, dt \;-\; \int_x^{\infty} e^{-t}\, t^{a-1}\, dt \\ &= \Gamma(a) \;-\; \Gamma(a,x) \end{aligned}$$

and by Example (4.5.2)

$$\gamma(a,x) \approx \Gamma(a) - e^{-x}\, x^{a-1} \sum_{n=0}^{\infty} (-1)^n (1-a)_n\, x^{-n} \ .$$

as $x \to \infty$. Although this series is divergent, it gives a good approximation only by summing few terms as shown in Table III. Following the same way as in Example I to find the optimal

N for a given x , $|a-N|$ equals the largest integral part of x .

Notice that if a is an integer the series terminates and the series gives the exact value of $\gamma(a,x)$.

TABLE II

a=1.05

n	x=.5	x=5	x=7
1	0.342172444	-8.056108470	-18.99702450
2	0.361966193	14.152879700	42.978363000
3	0.359481752	-13.72259040	-65.92488100
4	0.359730780	14.221877100	86.916931200
5	0.359709978	-9.103668210	-91.69326780
6	0.359711468	7.577121730	87.128021200
7	0.359711349	-2.857620240	-69.47938540
8	0.359711349	2.943460460	52.410522500
9	0.359711349	0.041315541	-32.95956420
10	0.359711349	1.361067770	21.391418500
11		0.810964584	-10.32528020
12		1.022609710	6.758397100
13		0.947001636	-1.785774230
14		0.972210288	2.202453610
15		0.964330733	0.457241237
16		0.966648579	1.175989150
17		0.966004491	0.896430373
18		0.966173768	0.999440372
19		0.966131151	0.963381648
20		0.966141164	0.975402117
21		0.966138542	0.971576512
22		0.966138899	0.972740412
23		0.966138542	0.972400606
24		0.966138542	0.972495317
25		0.966138542	0.972469568
26			0.972476006
27			0.972474158
28			0.972474158
29			0.972473681
30			0.972473681
31			0.972473681
32			0.972473681
33			0.972473681
34			0.972473681
35			0.972473681

TABLE III

a = 1.05

n	x=.5	x=5	x=7
1	0.32904291	0.966124415	0.972487748
2	0.44035857	0.966138244	0.972488701
3	0.00622767	0.966132879	0.972488463
4	2.56759930	0.966136038	0.972488582
5	-17.667221	0.966133535	0.972488523
6	182.657532	0.966136038	0.972488523
7	-2201.2082	0.966133058	0.972488523
8	30934.5234	0.966137230	0.972488523
9	-495923.56	0.966130674	0.972488523
10	8934835.00	0.966142416	0.972488582
11	-178737072	0.966118991	0.972488463
12	....	0.966170251	0.972488582
13	....	0.966047823	0.972488403
14	....	0.966364920	0.972488761
15	....	0.965480208	0.972487986
16	....	0.968125582	0.972489655
17	....	0.959686816	0.972485840
18	....	0.988294244	0.972495079
19	....	0.885593534	0.972471416
20	....	1.274828910	0.972535491

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